Roth's Theorems for Sets of Matrices

Robert M. Guralnick*

Department of Mathematics
University of Southern California
Los Angeles, California 90089-1113

Dedicated to H. Wielandt on his seventy-fifth birthday.

Submitted by Thomas J. Laffey

ABSTRACT

It is shown that Roth's theorems on the equivalence and similarity of block diagonal matrices hold for finite sets of matrices over a commutative ring.

Let R be a ring with 1. Denote by $R_{m\times n}$ the set of $m\times n$ matrices over R, and set $R_n = R_{n\times n}$. Gustafson [4] extended a result of Roth [9] over fields and proved:

THEOREM A. *Let R be commutative.*

(i) *If $A\in R_{m\times r}$, $B\in R_{s\times n}$, and $C\in R_{m\times n}$, then there exist $X\in R_{r\times n}$ and $Y\in R_{m\times s}$ such that*

$$AX - YB = C$$

if and only if

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

are equivalent.

(ii) *If $A\in R_m$, $B\in R_n$, and $C\in R_{m\times n}$, then there exists $X\in R_{m\times n}$ such that*

$$AX - XB = C$$

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if and only if

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

are similar.

There have been many proofs of this theorem for R a field (cf. [1, 7, 9]), mostly based on the existence of canonical forms. These proofs obviously will not extend to more general situations. However, Gustafson [4] used an ingenious idea to reduce the problem (for R commutative) to artinian rings, and then used a counting argument (which gives a simpler proof even for R a field). There are both postive and negative results for noncommutative rings (see [2, 3, 5, 6]).

Taussky recently asked whether Roth's similarity theorem can be extended to sets of matrices or equivalently to representations of finitely generated R-algebras. The purpose of this note is to show that both the similarity and equivalence theorems hold for sets of matrices over commutative rings. The main idea of the proof is to reduce to the case of a single pair of matrices over a bigger ring. The canonical form arguments over fields will not give the result, since there are no canonical forms for pairs of matrices. It is also interesting to note that by considering sets of matrices, the similarity problem becomes a special case of the equivalence problem.

Let $R[x_1,\dots,x_n]$ denote the polynomial ring over R where the x_i's commute with R and each other. If $\{C_i|1\leqslant i\leqslant r\}$ and $\{D_i|1\leqslant i\leqslant r\}$ are sets of $m\times n$ matrices over R, write $\{C_i\}\sim\{D_i\}$ if $UC_iV=D_i$, $1\leqslant i\leqslant r$, for some $U\in \mathrm{GL}_m(R)$ and $V\in\mathrm{GL}_n(R)$. If $m=n$, write $\{C_i\}\approx\{D_i\}$ if $UC_iU^{-1}=D_i$, $1\leqslant i\leqslant r$, for some $U\in\mathrm{GL}_n(R)$.

LEMMA 1.

(a) *If* $\{C_i|1\leqslant i\leqslant r\}\cup\{D_i|1\leqslant i\leqslant r\}\subseteq R_{m\times n}$, *then* $\{C_i\}\sim\{D_i\}$ $\Leftrightarrow$ Σx_iC_i *and* Σx_iD_i *are equivalent over* $R[x_1,\dots,x_r]$.

(b) *If* $m=n$, *set* $C_0=D_0=I_n$. *Then*

$$\{C_i|0\leqslant i\leqslant r\}\sim\{D_i|0\leqslant i\leqslant r\} \quad\Leftrightarrow\quad \{C_i|1\leqslant i\leqslant r\}\approx\{D_i|1\leqslant i\leqslant r\}.$$

(c) *If* $m=n$, *then* $\{C_i\}\approx\{D_i\}$ $\Leftrightarrow$ Σx_iC_i *and* Σx_iD_i *are similar over* $R[x_1,\dots,x_r]$.

Proof. (a): If $\{C_i\}\sim\{D_i\}$, the result is obvious. Conversely, suppose $U\tilde{C}=\tilde{D}V$ for some $U\in\mathrm{GL}_m(R[x_1,\dots,x_r])$ and $V\in\mathrm{GL}_n(R[x_1,\dots,x_r])$, where $\tilde{C} = \Sigma x_i C_i$ and $\tilde{D} = \Sigma x_i D_i$. Write $U = U_0 + \cdots + U_d$ and $V = V_0 + \cdots + V_d$, where U_i and V_i are homogenous of degree i. Equating degree one terms, we see that $U_0 C_i = D_i V_0$, $1 \leqslant i \leqslant r$. Moreover $U_0(U^{-1})_0 = I_m$ and $V_0(V^{-1})_0 = I_n$, so U_0 and V_0 are invertible over R, and the result follows.

Now (b) is obvious, and (c) follows either from (a) and (b), or exactly as in the proof of (a). ∎

THEOREM B. *Let*

$$M_i = \begin{pmatrix} A_i & C_i \\ 0 & B_i \end{pmatrix} \quad \textit{and} \quad N_i = \begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix},$$

$1 \leqslant i \leqslant t$, *where* $A_i \in R_{m\times r}$, $B_i \in R_{s\times n}$, *and* $C_i \in R_{m\times n}$.

(i) *If* $R[x_1, \ldots, x_t]$ *satisfies Roth's equivalence property (Theorem* A(i)*), then* $\{M_i\} \sim \{N_i\} \Leftrightarrow$ *there exist* $X \in R_{r\times n}$ *and* $Y \in R_{m\times s}$ *such that* $C_i = A_i X - Y B_i$, $1 \leqslant i \leqslant r$.

(ii) *If* $R[x_1, \ldots, x_t]$ *satisfies Roth's similarity property (Theorem* A(ii)*) and* $m = r$ *and* $n = s$, *then* $\{M_i\} \approx \{N_i\} \Leftrightarrow C_i = A_i X - X B_i$, $1 \leqslant i \leqslant r$, *for some* $X \in R_{m\times n}$.

(iii) *If* R *satisfies the simultaneous Roth equivalence property (the conclusion of Theorem* B(i)*) for sets of* $t+1$ *matrices, then* R *satisfies the simultaneous Roth similarity property (the conclusion of Theorem* B(ii)*) for sets of* t *matrices.*

Proof. (i): If X and Y exist, set

$$U = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Then U and V are invertible and $UM_i = N_i V$, $1 \leqslant i \leqslant r$. For the converse, we just apply Lemma 1(a) and Theorem A(i), to obtain $\tilde{C} = \tilde{A}\tilde{X} - \tilde{Y}\tilde{B}$ over $R[x_1, \ldots, x_t]$ where $\tilde{A} = \Sigma x_i A_i$, $\tilde{B} = \Sigma x_i B_i$, and $\tilde{C} = \Sigma x_i C_i$. By equating degree one terms, we see that $C_i = A_i X - Y B_i$, $1 \leqslant i \leqslant r$, where X and Y are the degree zero terms of $\tilde{X}$ and $\tilde{Y}$.

Now (ii) follows in exactly the same manner. Also (iii) follows by Lemma 1(c). ∎

COROLLARY. *If* R *is a module finite algebra over a commutative ring, then* R *satisfies the simultaneous Roth similarity and equivalence property.*

Proof. By [5], $R[x_1,\dots,x_t]$ satisfies the Roth properties. Thus the theorem implies the result. ∎

It should be possible to replace t by $t-1$ in Theorem B. This is true for $t=1$ (see [2]). It is still an open question whether an infinite dimensional (over its center) division ring satisfies any of these results (except for the equivalence property of Theorem A).

We now show how the similarity result of the Corollary can be derived from a result of Miyata [8] (independently of Gustafson's theorem). For convenience, we shall assume our ring is commutative instead of a module finite algebra. The crucial idea of the proof is to observe that the existence of a solution to the matrix equations $C_i = A_iX - XB_i$, $1 \leqslant i \leqslant r$, is equivalent to the existence of a common M_i-invariant complement to the M_i-invariant subspace corresponding to the A_i.

PROPOSITION C. *Let R be a commutative ring. Suppose $A_i \in R_m$, $B_i \in R_n$, and $C_i \in R_{m\times n}$ for $1 \leqslant i \leqslant r$. Set*

$$M_i = \begin{pmatrix} A_i & C_i \\ 0 & B_i \end{pmatrix} \quad \text{and} \quad N_i = \begin{pmatrix} A_i & 0 \\ 0 & B_i \end{pmatrix}.$$

Then $\{M_i\} \approx \{N_i\}$ $\Leftrightarrow$ there exists $X \in R_{m\times n}$ such that $C_i = A_iX - XB_i$, $1 \leqslant i \leqslant r$.

Proof. We can assume R is finitely generated as a ring and hence is noetherian. Let $S = R\langle x_1,\dots,x_r\rangle$ be the free R-algebra on r variables (i.e., each x_i commutes with R, but no other relations are assumed). Let P denote the space of column vectors of length $m+n$ over R. Then we obtain two representations α and β of S into R_{m+n} (or equivalently two S-module structures, P_α and P_β, on P) by defining

$$\alpha(x_i)v = M_iv \quad \text{and} \quad \beta(x_i)v = N_iv.$$

Now $\{M_i\} \approx \{N_i\}$ is exactly the statement that α and β are equivalent representations (or $P_\alpha \cong P_\beta$). So $K = \ker\alpha = \ker\beta$. Then $\bar{S} = S/K$ is a module finite R-algebra. Let N_α (N_β) denote the $\bar{S}$-submodules of P_α (P_β) consisting of the column vectors whose last n terms are zero. Thus $P_\alpha \cong P_\beta \cong N_\beta \oplus (P_\beta/N_\beta) \cong N_\alpha \oplus (P_\alpha/N_\alpha)$. By Miyata's theorem [8], this implies N_α is a direct summand of P_α. Thus there exists an idempotent E in $\mathrm{Hom}_S(P_\alpha, P_\alpha)$ with the image of E being N_α. Written in block matrix form,

$$E = \begin{pmatrix} I_m & X \\ 0 & 0_n \end{pmatrix}.$$

Since $EM_i = M_iE$ for each i, it follows that $C_i = A_iX - XB_i$, $1 \leqslant i \leqslant r$. ■

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